Main terms in moments

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Classical Mean-Value theorems for ζ

Hardy and Littlewood (1918):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 \, dt \sim \log T$$

Ingham (1926):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 \, dt \quad \sim \quad 2\prod_p (1-\frac{1}{p})(1+\frac{1}{p})\frac{\log^4 T}{4!}$$

At the Amalfi conference in 1989:

Theorem : (Conrey and Ghosh)

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^6 \, dt \geq 10.13 \; a_3 \; \frac{\log^9 T}{9!}$$

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{p^j}$$

Conrey and Ghosh conjecture: 1992

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^6 \, dt \sim 42 \prod_p \left(1-\frac{1}{p}\right)^4 \left(1+\frac{4}{p}+\frac{1}{p^2}\right) \frac{\log^9 T}{9!}$$

Conrey and Gonek conjecture: 1998

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^8 \, dt \sim 24024 \prod_p \left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right) \frac{\log^{16} T}{16!}$$

Keating and Snaith formula:

$$\int_{U(N)} |\det(I-U)|^{2k}\, dU \quad = \quad \frac{(N+1)(N+2)^2\ldots(N+k)^k(N+k+1)^{k-1}\ldots(N+2k-1)}{1\cdot 2^2\cdot\cdots\cdot k^k\cdot(k+1)^{k-1}\cdot\cdots\cdot(2k-1)}$$

$$\sim \quad g_k\frac{N^{k^2}}{k^2!}$$

$$g_k = \frac{k^2!}{1\cdot 2^2\cdot\cdots\cdot k^k\cdot(k+1)^{k-1}\cdot\cdots\cdot(2k-1)}$$

$$g_1 = 1 \qquad g_2 = 2 \qquad g_3 = 42 \qquad g_4 = 24024$$

$$g_5 = 701149020$$

Conjecture (KS):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\ dt \sim g_k a_k \frac{\log^{k^2} T}{k^2!}$$

Theorem:

$$\int_0^T |\zeta(1/2+it)|^4 \ dt$$
$$= \int_0^T P_2\left(\log \frac{t}{2\pi}\right) \ dt + O(T^{2/3+\epsilon})$$

where

$$\begin{aligned}
P_2(x) \ = \ & \frac{1}{2\pi^2}x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right)x^3 \\
& + \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right)x^2 \\
& + \frac{12}{\pi^8}(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 \\
& \qquad + 288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4)x \\
& + \frac{4}{\pi^{10}}(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4 \\
& \qquad + 3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 \\
& \qquad + 3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 \\
& \qquad - 864\gamma_1\zeta'(2)^2\pi^4 + 5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 \\
& \qquad - 15552\zeta''(2)\zeta'(2)^2\pi^2 - 96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4),
\end{aligned}$$

Conjecture (CFKRS) The Recipe

Let A and B be sets of small complex numbers and $\prod_{\alpha\in A}\zeta(s+\alpha)=\sum_{n=1}^{\infty}\frac{\tau_A(n)}{n^s}$

Then, with s=1/2+it and $\psi\in C^\infty[1,2]$

$$\int \psi\left(\frac{t}{T}\right)\prod_{\alpha\in A}\zeta(s+\alpha)\prod_{\beta\in B}\zeta(1-s+\beta)\ dt$$

$$=\int \psi\left(\frac{t}{T}\right)\sum_{\substack{U\subset A, V\subset B\\ |U|=|V|}}\left(\frac{t}{2\pi}\right)^{-(U+V)}\mathcal{B}_{\overline{U}\cup V^-,\overline{V}\cup U^-}(1)\ dt+O(T^{1-\delta})$$

where

$$\mathcal{B}_{A,B}(s)=\sum_{n=1}^{\infty}\frac{\tau_A(m)\tau_B(m)}{m^s}$$

Note that each term has a total of |A| |B| singularities; but the sum is analytic .

Conjecture (C, Farmer, Keating, Rubinstein, Snaith)

$$\int_0^T |\zeta(1/2+it)|^6 \, dt$$
$$= \int_0^T P_3(\log \frac{t}{2\pi}) \, dt + O(T^{1/2+\epsilon})$$

where

$$\begin{aligned} P_3(x) \;=\;& 0.000000570852703465278839837684144525231 3\,x^9 \\ &+ 0.00040502133088411440331215332025984\,x^8 \\ &+ 0.011072455215246998350410400826667\,x^7 \\ &+ 0.14840073080150272680851401518774\,x^6 \\ &+ 1.0459251779054883439385323798059\,x^5 \\ &+ 3.9843850948235347247479640734 29\,x^4 \\ &+ 8.6073191457812067561483476362 9\,x^3 \\ &+ 10.274330830703446134183009522\,x^2 \\ &+ 6.5939130206497581046571339 2\,x \\ &+ 0.91651550763789305901785 43. \end{aligned}$$

RMT analogue

Theorem (CFKRS). Let

$$Z(A,B) = \prod_{\substack{\alpha\in A\\ \beta\in B}} z(\alpha+\beta)$$

where $z(x) = (1-e^{-x})^{-1}$. Then

$$\int_{U(N)} \prod_{\alpha\in A} \Lambda_X(e^{-\alpha}) \prod_{\beta\in B} \Lambda_{X^*}(e^{-\beta})\; dX$$

$$= \sum_{\substack{S\subset A\\ T\subset B\\ |S|=|T|}} e^{-N(S+T)} Z(\overline{S}\cup T^-, \overline{T}\cup S^-)$$

where

$$\overline{S} = A - S \qquad -S = \{-s : s\in S\} \qquad \sum s = \sum_{s\in S} s$$

This matches perfectly with the recipe!

Conrey, Iwaniec, Soundararajan proved a sixth moment estimate for Dirichlet L-functions which found the full 9th degree polynomial above.

Chandee and Li obtained the leading order term (the 24024) assuming RH for the 8th moment of this family.

Nathan Ng proved that the sixth moment of zeta with all lower order terms can be obtained from precise information about the shifted divisor problem.

Combinatorics of main terms

There is interest in how new main terms enter the picture in moment problems as the order of the moment grows. Classically we can do the second moment of zeta by diagonal analysis; but the fourth moment requires shifted convolution sums.

For averages of quadratic L-functions one uses diagonal analysis for the first and second moment and then Soundararajan's Poisson formula for quadratic characters for the third moment. New main terms arise from the "square" values of k after using Poisson.

For averaging cusp form L-functions at the center via the Peterson formula one initially uses only the diagonal terms at the start; then Kowalski, Michel and Vanderkaam show how to use parts of the Kloosterman sum to obtain some off-diagonal contributions; further investigation leads to off-off-diagonal contributions to the main term.

In the asymptotic large sieve applications of Conrey, Iwaniec and Soundararajan (eg for the sixth moment of Dirichlet L-functions) the final main terms seem to be located in a remote part of the complex plane, far from other contributing singularities.

In Zhang and Diaconu using multiple Dirichlet series found a polar term at 3/4!

Moments of long Dirichlet polynomials

$$\int_0^\infty \psi\left(\frac{t}{T}\right) \sum_{m\leq X} \frac{a_m}{m^{1/2+it}} \sum_{n\leq X} \frac{b_n}{n^{1/2-it}}\ dt$$

$$= T\hat{\psi}(0) \sum_{n\leq X} \frac{a_n b_n}{n}$$

$$+T \sum_{m\neq n} \frac{a_m b_n}{\sqrt{mn}} \hat{\psi}\left(\frac{T}{2\pi} \log \frac{n}{m}\right)$$

The off-diagonal piece is

$$\sum_{\substack{h\neq 0\\ 0<m+h\leq X}} \frac{a_m b_{m+h}}{\sqrt{m(m+h)}} \hat{\psi}\left(\frac{T}{2\pi}\log\left(1+\frac{h}{m}\right)\right)$$

We can often rewrite this as

$$\sim \sum_{\substack{h\neq 0\\ 0<m\leq X}} \frac{a_m b_{m+h}}{m} \hat{\psi}\left(\frac{Th}{2\pi m}\right)$$

The case of t-aspect for Ramanujan tau L-function

How does one average the moments of a cusp form L-function in t-aspect?

The shifted convolutions play a role.

$$\tau^*(n) = \tau(n)n^{-11/2}$$

Good (1983)

$$S(X,h) := \sum_{n\leq X} \tau^*(n)\tau^*(n+h)$$

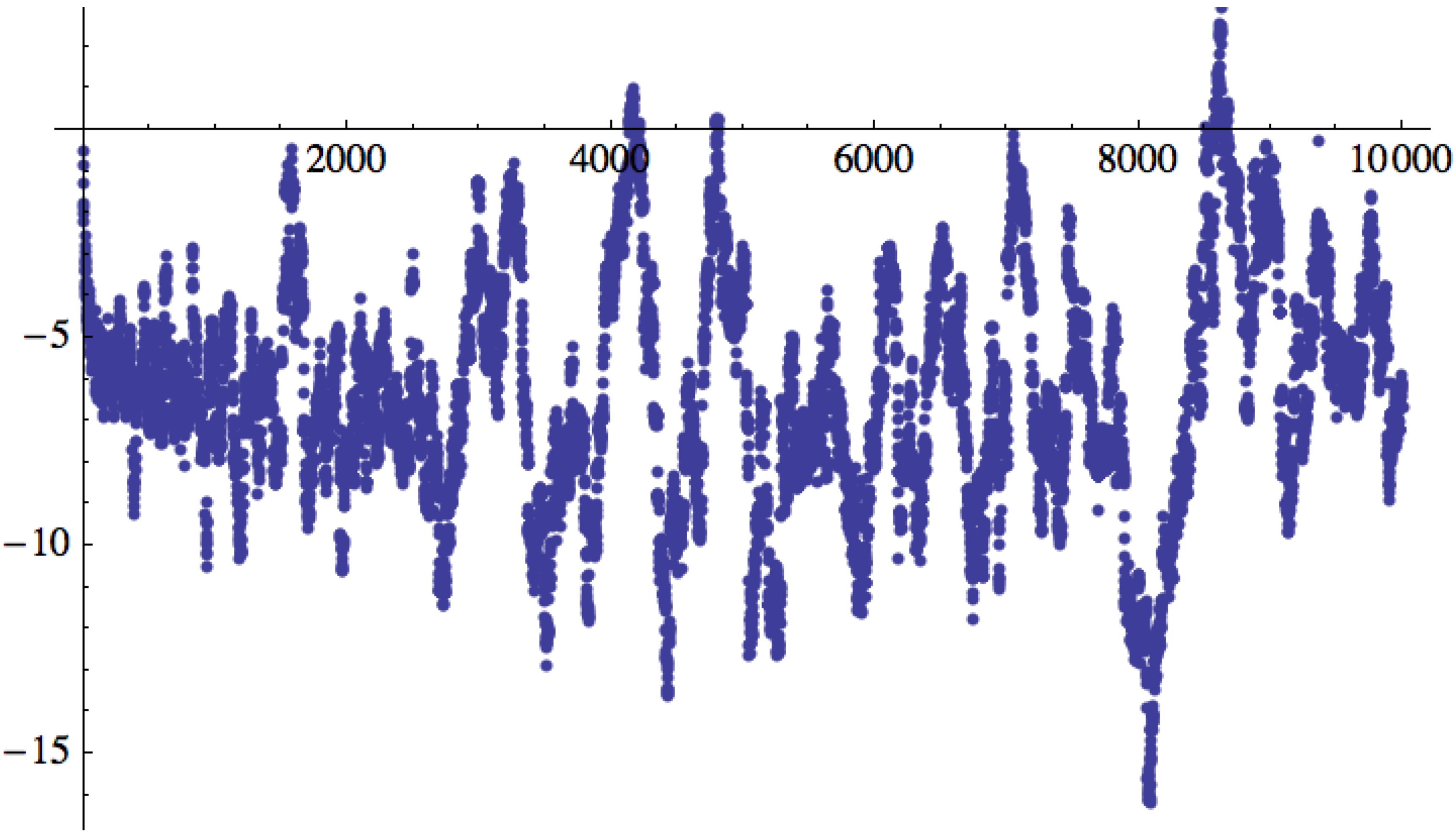
$$S(X,h) \ll_h X^{2/3}$$

$$\frac{1}{\sqrt{X}} \sum_{h\leq\sqrt{X}} |S(X,h)| \ll X^{1/2+\epsilon}$$

Blomer (2005)

$$\sum_{n \leq X} \tau^*(n)\tau^*(n+h) \ll X^{1-\delta}$$

uniformly for $h \ll X^{2-\eta}$

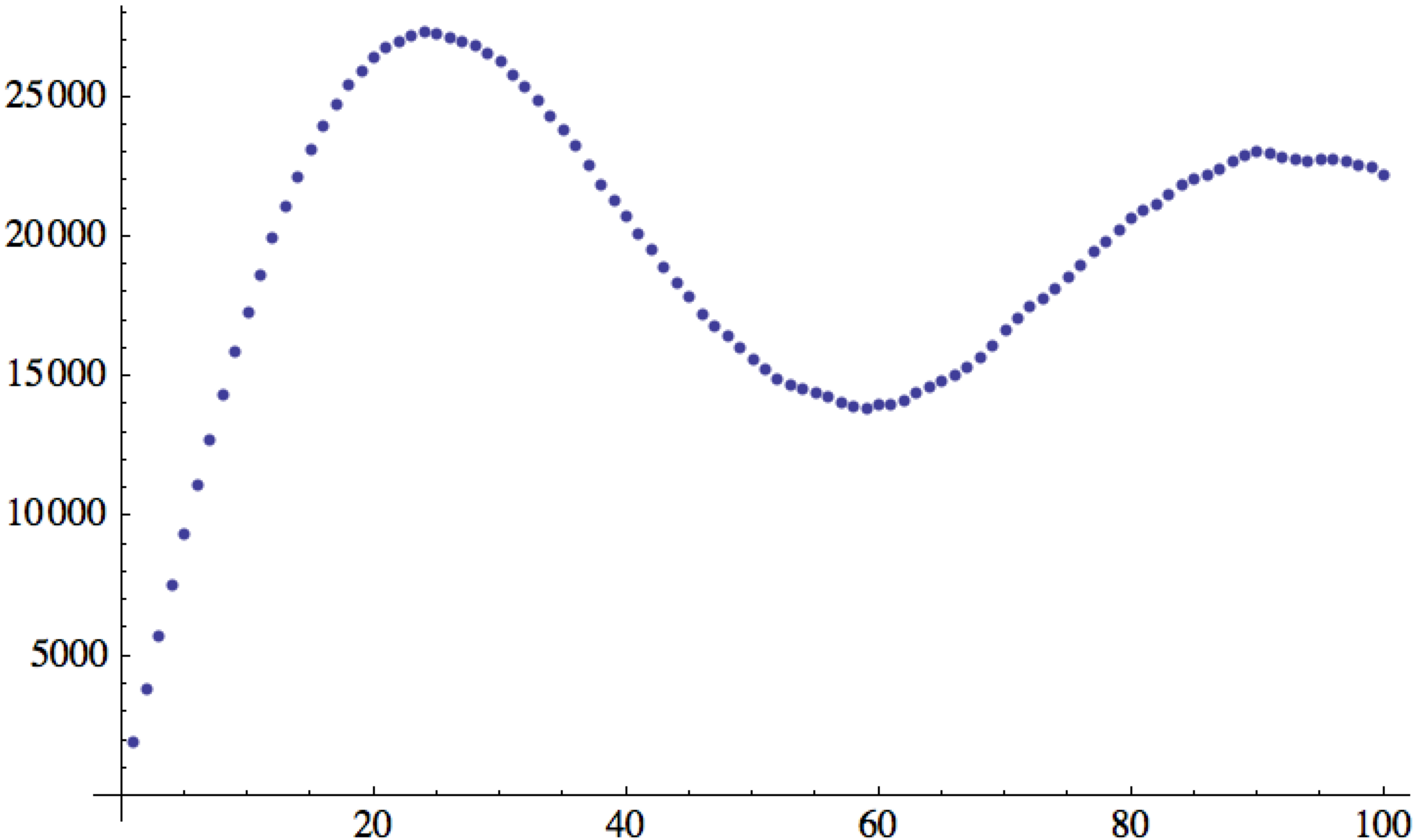


S(X,1) for 1< X < 10000

Good:

$$T(X;H) := \sum_{n\le X} \left| \sum_{h\le H} \tau^*(n+h) \right|^2 = CXH + O(X^{2/3+\epsilon} H^{5/3})$$

uniformly for $H \ll X^{1/2}$



T(X;H) for X = 5000; 1< H < 100

Averaging a long tau polynomial

$$\int \psi\left(\frac{t}{T}\right)\left|\sum_{n\le X}\frac{\tau^*(n)}{n^{1/2+it}}\right|^2 \, dt$$

$$= T\hat{\psi}(0)\sum_{n\le X}\frac{\tau^*(n)}{n} + 2T\sum_{\substack{h\neq 0\\ T\le m\le X}}\frac{\tau^*(m)\tau^*(m+h)}{m}\hat{\psi}\left(\frac{Th}{2\pi m}\right) + o(T)$$

$$\sim \begin{cases} T\hat{\psi}(0)\sum_{n\le X}\dfrac{\tau^*(n)^2}{n} & X \ll T^2 \\ \int \psi\left(\frac{t}{T}\right)\left|L_{\tau^*}(1/2+it)\right|^2 \, dt & X \gg T^2 \end{cases}$$

But

$$\int \psi\left(\frac{t}{T}\right)\left|L_{\tau^*}(1/2+it)\right|^2 \, dt \sim T\hat{\psi}(0)\sum_{n\le T^2}\frac{\tau^*(n)^2}{n}$$

So, it must be the case that

$$2 \sum_{\substack{h \neq 0 \\ T \leq m \leq X^2}} \frac{\tau^*(m)\tau^*(m+h)}{m} \hat{\psi}\left(\frac{Th}{2\pi m}\right) \sim C\hat{\psi}(0) \log \frac{T^2}{X}$$

for $X \gg T^2$

where C is such that

$$\sum_{n \leq X} \frac{\tau^*(n)^2}{n} \sim C \log X$$

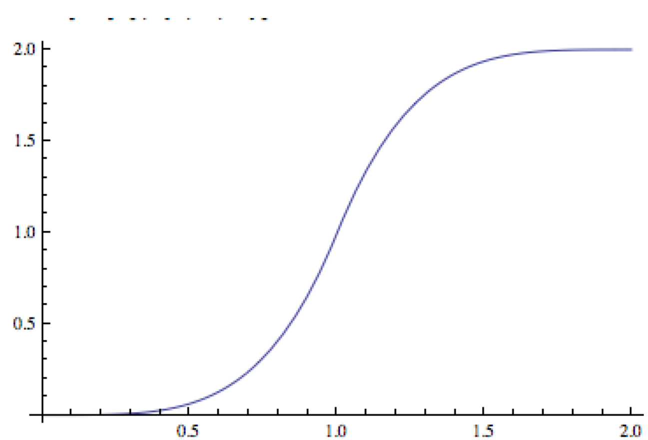
The case of zeta

Zeta-polynomials

$$\frac{1}{T}\int_0^T \left| \sum_{n\le X} \frac{d_k(n)}{n^s} - \mathrm{Res}_{w=1-s} \frac{\zeta(s+w)^k X^w}{w} \right|^2 dt \sim M_k(\alpha)\ a_k\ \frac{\log^{k^2} T}{k^2!}$$

where $s = 1/2 + it$ and $X = T^\alpha$. For example

$$M_2(\alpha) = \begin{cases} \alpha^4 & \text{if } 0<\alpha<1 \\ -\alpha^4 + 8\alpha^3 - 24\alpha^2 + 32\alpha - 14 & \text{if } 1<\alpha<2 \\ 2 & \text{if } \alpha > 2 \end{cases}$$



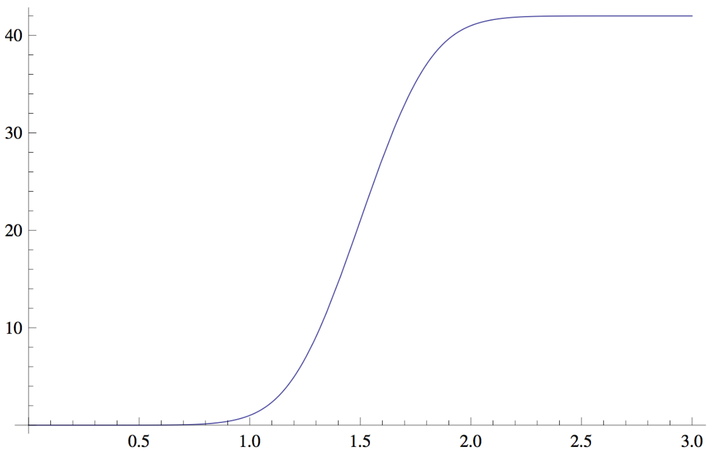


FIGURE 2. The plot of $M_3(\alpha)$ for $0 < \alpha < 3$.

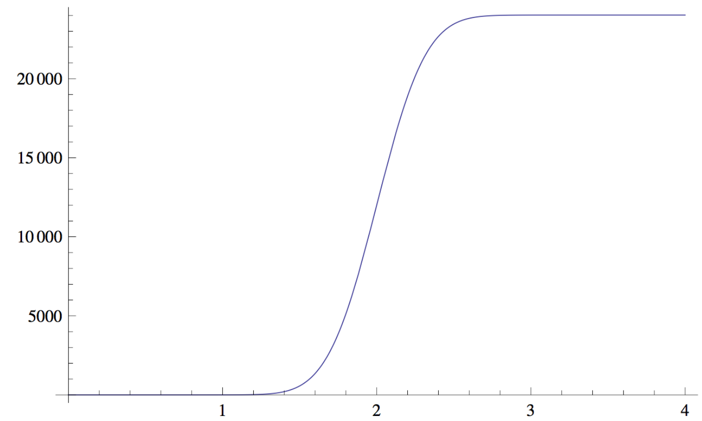


FIGURE 3. The plot of $M_4(\alpha)$ for $0 < \alpha < 4$.

Let

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

and

$$\Lambda_U(s) = \sum_{j=0}^{N} Sc_j(U) s^j$$

To what extent do the coefficients of $\Lambda_U(s)^k$ behave like $d_k(n)n^{it}$?

Diaconis - Gamburd formula

If $j_1 + \cdots + j_k = h_1 + \cdots + h_\ell \leq N$, then

$$\int_{U(N)} Sc_{j_1}(U) \ldots Sc_{j_k}(U) \overline{Sc_{h_1}(U) \ldots Sc_{h_\ell}(U)} \, du$$

is the number of $k \times \ell$ matrices with non-negative integer entries and row sums $j_1, \ldots, j_k$ and column sums $h_1, \ldots h_\ell$.

For example,

$$\int_{U(N)} Sc_j(U) \overline{Sc_k(U)} \, du = 0 \qquad\qquad \int_{U(N)} |Sc_j(U)|^2 \, du = 1$$

and

$$\int_{U(N)} |Sc_j(U) Sc_k(U)|^2 \, dU = 1 + \min\{j, k\}$$

if $j \neq k$ and $j + k \leq N$.

Conjecture

$$\frac{1}{T(\log T)^{k^2}} \int_0^T \left| \sum_{m \le T^\alpha} \frac{d_k(m)}{m^{1/2+it}} - \mathrm{Res}_{s=1} \frac{\zeta(s)^k T^{\alpha s}}{s} \right|^2 dt$$

$$\sim \frac{a_k}{N^{k^2}} \int_{U(N)} \left| \sum_{\substack{j_1+\cdots+j_k \le \alpha N \\ j_i \le N}} Sc_{j_1} \ldots Sc_{j_k} \right|^2 dU$$

Sandro Bettin assumes the recipe and proves this.

By orthogonality

$$\int_{U(N)} \left| \sum_{j_1+\cdots+j_k \le \alpha N} Sc_{j_1} \ldots Sc_{j_k} \right|^2 du = \sum_{m \le \alpha N} I_k(m, N)$$

where

$$I_k(m; N) := \int_{U(N)} \left| \sum_{\substack{j_1+\cdots+j_k = m \\ j_i \le N}} Sc_{j_1}(U) \ldots Sc_{j_k}(U) \right|^2 dU$$

Keating, Rodgers, Roddity-Gershon, Rudnick formula

If m < N then

$$I_k(m;N) = \binom{k^2-1+m}{m}$$

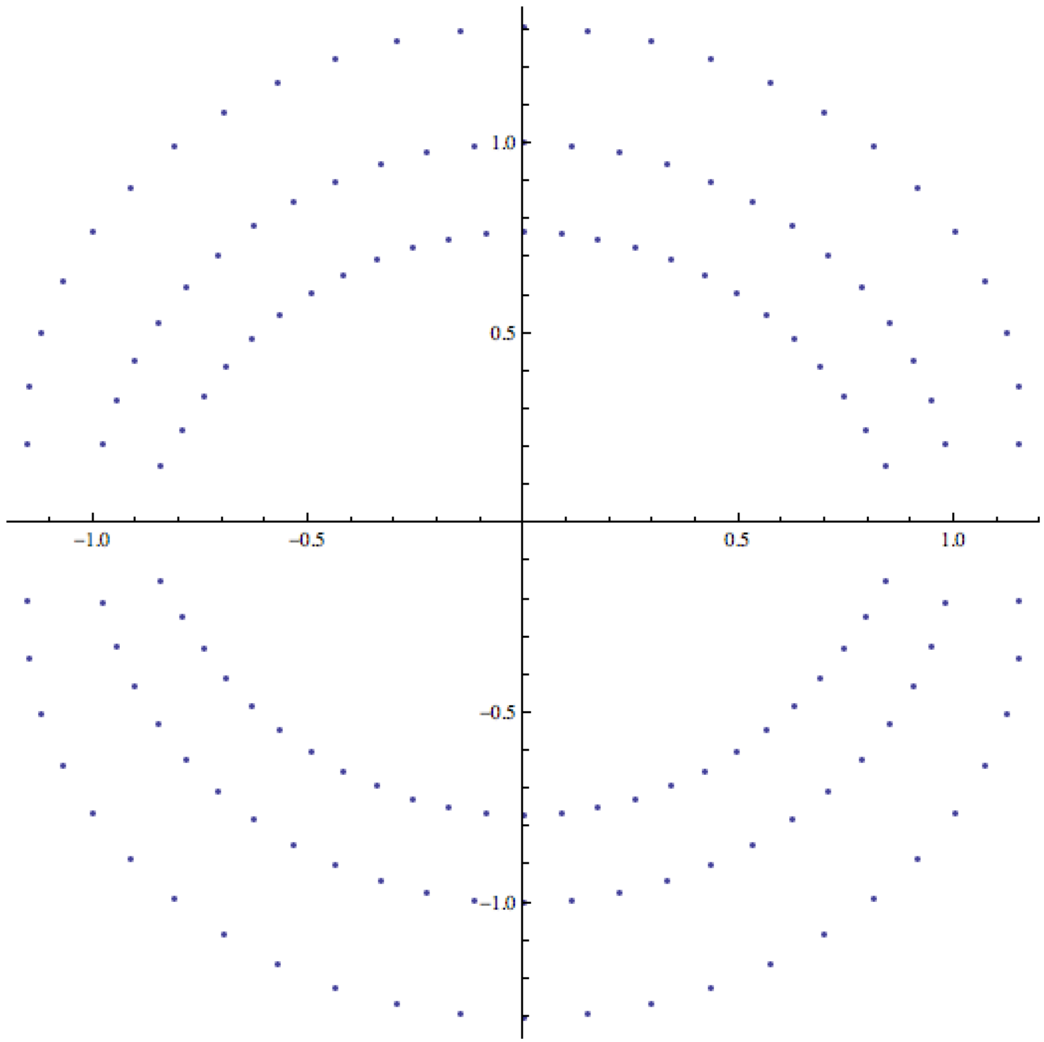
By the functional equation this also works for (k-1)N , m < kN.

It's not so clear what the formula looks like when N < m < 2n.

Keating, Rodgers, Roddity-Gershon, Rudnick formula

$I_k(m;N)$ is equal to the number of $k \times k$ matrices with non-negative integer entries at most N in size whose rows are weakly increasing, columns are weakly decreasing and whose anti-diagonal sums to $kN - m$

(Gelfond-Tsetlin patterns)



Zeros of $\int_{U(25)} \Lambda_U(x)^3 \Lambda_{U^*}(x)^3 \; dU = \sum_{m=0}^{25} I_k(m, 25) x^{2m}$

Keating, Rodgers, Roddity-Gershon, Rudnick

$$I_k(m;N) = \gamma_k(c) N^{k^2-1} + O(N^{k^2-2})$$

$$c = m/N$$

where

$$\gamma_k(c) = \frac{1}{k! G(1+k^2)} \int_{[0,1]^k} \delta_c(w_1 + \cdots + w_k) \Delta(w_1, \ldots, w_k)^2 dw$$

G is the Barnes function.

$$\gamma_k(c) = \frac{M_k'(c)}{k^2!}$$

Conjecture: Keating, Rodgers, Roddity-Gershon, Rudnick;
Rodgers, Soundararajan

$$\frac{1}{X}\int_X^{2X}\left|\sum_{x\le n\le x+H} d_k(n)\right|^2 dx - \left(\frac{1}{X}\int_X^{2X}\sum_{x\le n\le x+H} d_k(n)\ dx\right)^2$$

$$\sim a_k(q)\gamma_k(c)H\left(\log\frac{X}{H}\right)^{k^2-1}$$

with $\dfrac{\log X}{\log \frac{X}{H}} \to c \in (0,k)$

Lester has made progress on the divisor problem in short intervals.
Keating, Rodgers, Roddity-Gershon, Rudnick prove the function field analogue of this.

Conjecture: Keating, Rodgers, Roddity-Gershon, Rudnick;
Rodgers, Soundararajan

$$\sum_{\substack{1\le a\le q\\(a,q)=1}} \left| \sum_{\substack{n\equiv a \bmod q\\ n\le X}} d_k(n) - \frac{1}{\phi(q)} \sum_{\substack{(n,q)=1\\ n\le X}} d_k(n) \right|^2$$

$$\sim a_k(q)\gamma_k(c)H(\log X)^{k^2-1}$$

with $\frac{\log X}{\log q} \to c \in (0,k)$

Rodgers and Soundararajan prove this for delta<c<2-delta (assuming GRH).
Keating, Rodgers, Roddity-Gershon, Rudnick prove a function field analogue of this.

Basor, Ge, Rubinstein

$$\gamma_k(c) = \frac{1}{k!G(1+k^2)} \int_{[0,1]^k} \delta_c(w_1 + \cdots + w_k)\Delta(w_1, \ldots, w_k)^2 dw$$

$$\gamma_k(c) = \frac{1}{G(1+k)^2} \int_{-\infty}^{\infty} \exp(2\pi i u c) D_k(2\pi i u)\ du$$

$$D_k(t) = \det_{k\times k}\left(g^{(i+j-2)}(u)\right)$$

$$g(u) = \int_0^1 \exp(-tx)\ dx$$

$$D_k(t) = D_k(0)\exp\left(\sum_{m=1}^{\infty}\frac{c_m(k)}{m}t^m\right)$$

D satisfies a Painleve equation which leads to a recursion formula:

$$c_M(k) = \frac{1}{(M-1)(M-2)}\sum_{m=0}^{M-3}(m+2)c_{m+2}(k)\left(c_{M-m-2}(k-1)+c_{M-m-2}(k+1)-2c_{M-m-2}(k)\right)$$

$$c_1(k) = -\frac{k}{2}$$

$$c_2(k) = \frac{k^2}{4(4k^2-1)}$$

Back to moments of zeta

Divisor correlations

We need information about

$$\sum_{n \leq X} \tau_A(n)\tau_B(n+h)$$

The delta method of Duke, Friedlander and Iwaniec (1993) can provide the needed conjecture.

Bill Duke

John Friedlander

Henryk Iwaniec

Delta method conjecture

$$\langle \tau_A(m)\tau_B(m+h)\rangle_{m=u}$$
$$\sim \sum_{q=1}^{\infty} r_q(h)\langle \tau_A(m)e(m/q)\rangle_{m=u}\langle \tau_B(n)e(n/q)\rangle_{n=u}$$

$$\langle \tau_A(m)e(m/q)\rangle_{m=u} = \frac{1}{2\pi i}\int_{|w-1|=\epsilon} D_A(w,e(1/q))u^{w-1}\,dw$$

where

$$D_A(w,e(1/q)) = \sum_{n=1}^{\infty} \frac{\tau_A(n)e(n/q)}{n^w}$$

$r_q(h)$ is the Ramanujan sum and D_A is the Estermann function

The poles of this Dirichlet series can be determined by replacing the exponential by Dirichlet characters and finding the coefficient of the trivial character (i.e. zeta).

Assuming delta-conjecture

$$\int_0^\infty \psi(\frac{t}{T}) \sum_{\substack{m\le T^r\\ n\le T^r}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it}\ dt$$

$$= \int_0^\infty \psi(\frac{t}{T})\Bigg(\mathcal{B}_{A,B}(1;T^r) + \sum_{\substack{\alpha\in A\\ \beta\in B}} \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \mathcal{B}_{A',B'}(1;T^r)\Bigg)$$

$+O(T^{1-\delta})$ where $1 \le r < 2$ and

$$A' = A - \{\alpha\} \cup \{-\beta\}, B' = B - \{\beta\} \cup \{-\alpha\}$$

This relies on the identity

$$\operatorname*{Res}_{\substack{w=1-\alpha \\ z=1-\beta}} \sum_{q=1}^{\infty} \mu(q) \sum_{d=1}^{\infty} d^{z+w-1} \zeta(w+z-1) \mathcal{D}_A\left(w, e(-\frac{1}{qd})\right) \mathcal{D}_B\left(z, e(-\frac{1}{qd})\right)$$
$$= \mathcal{B}(A' \cup \{-\beta\}, B' \cup \{-\alpha\}).$$

where

$$D_A(s, e(\frac{1}{q})) := \sum_{m=1}^{\infty} \frac{\tau_A(m) e(\frac{m}{q})}{m^s}$$

and

$$\mathcal{B}(A, B) = \sum_{n=1}^{\infty} \frac{\tau_A(n) \tau_B(n)}{n}$$

Equating Euler products, the identity is:

Suppose $\hat{\alpha} \in A$ and $\hat{\beta} \in B$.
Let $A' = A - \{\hat{\alpha}\}$ and $B' = B - \{\hat{\beta}\}$. Then

$$\sum_{j=0}^{\infty} \frac{\tau_{A' \cup \{-\hat{\beta}\}}(p^j)\tau_{B' \cup \{-\hat{\alpha}\}}(p^j)}{p^j}$$

$$= \sum_{h=0}^{\infty} \frac{1}{p^{h(1-\hat{\alpha}-\hat{\beta})}} \sum_{j=0}^{\infty} \frac{\tau_{A'}(p^j)}{p^{j(1-\hat{\alpha})}} \sum_{k=0}^{\infty} \frac{\tau_{B'}(p^k)}{p^{k(1-\hat{\beta})}} \sum_{d,q=0}^{\infty} \frac{\mu(p^q)}{p^{d+q(2-\hat{\alpha}-\hat{\beta})}} \sum_{m=0}^{\infty} \frac{\tau_{A'}(p^{m+d+q})}{p^{m(1-\hat{\alpha})}} \sum_{n=0}^{\infty} \frac{\tau_{B'}(p^{n+d+q})}{p^{n(1-\hat{\beta})}}$$

Identifying `geometric' factors and the location of the poles of both sides of the identity is a matter of equating the `linear' term of the Euler products; i.e. the value at p of the arithmetic function. So, in some sense the identification of the whole p-factor is more complex.

Thus, we see the `one-swap' terms arise from the standard shifted divisor problem.

Where are the rest of the terms from the recipe???

What is the RMT analogue of the input from the (averaged over h) delta method?

What if $r > 2$?

Say $\ell < r < \ell + 1$

To find the higher-swap terms we will need convolutions of shifted divisors …

Conrey - Keating approach

Split A and B up into $A = A_1 \cup \cdots \cup A_\ell$ and $B = B_1 \cup \cdots \cup B_\ell$. Then

$$\sum_{m,n<T^r} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} \hat{\psi}(\frac{T}{2\pi}\log\frac{m}{n})$$

is related to

$$\sum_{\substack{A=A_1\cup\cdots\cup A_\ell \\ B=B_1\cup\cdots\cup B_\ell}} \sum_{\substack{M_1\ldots M_\ell=N_1\ldots N_\ell \\ (M_i,N_i)=1}} \prod_{j=1}^{\ell} \left(\sum_{m_j,n_j} \frac{\tau_{A_j}(m_j)\tau_{B_j}(n_j)}{\sqrt{mn}} \right) \hat{\psi}(\frac{T}{2\pi}\log\frac{m_1\ldots m_\ell}{n_1\ldots n_\ell})$$

subject to $\prod m_i, \prod n_i \le T^r$ and

$$* \left\{ \begin{array}{ccc} M_1 m_1 & = & N_1 n_1 + h_1 \\ & \cdots & \\ M_\ell m_\ell & = & N_\ell n_\ell + h_\ell \end{array} \right\}$$

Note that

$$\hat{\psi}(\frac{T}{2\pi}\log\frac{m_1\ldots m_\ell}{n_1\ldots n_\ell}) \sim \hat{\psi}(\frac{T}{2\pi}\sum\frac{h_i}{n_i N_i})$$

which controls the ranges of the sums.

Delta method conjecture with general linear constraint

$$\langle \tau_A(m)\tau_B(n)\rangle_{m=u}^{(*)}$$
$$\sim \frac{1}{M}\sum_{q=1}^{\infty} r_q(h)\langle \tau_A(m)e(mN/q)\rangle_{m=u}\langle \tau_B(n)e(nM/q)\rangle_{n=\frac{uN}{M}}$$

where $(*) : mM - nN = h$

$$\langle \tau_A(m)e(mN/q)\rangle_{m=u} = \frac{1}{2\pi i}\int_{|w-1|=\epsilon} D_A(w, e(\frac{N}{q}))u^{w-1}\, dw$$

where

$$D_A(w, e(\frac{N}{q})) = \sum_{n=1}^{\infty} \frac{\tau_A(n)e(nN/q)}{n^w}.$$

The poles of this Dirichlet series can be determined by replacing the exponential by Dirichlet characters and finding the coefficient of the trivial character (i.e. zeta).

Connecting divisor correlations and the recipe

$$\frac{1}{2\pi i}\int_{(2)} X^s\Big(\frac{T}{2\pi}\Big)^{-\ell s}\sum_{\substack{(M_1,N_1)=\cdots=(M_\ell,N_\ell)=1\\ N_1\dots N_\ell=M_1\dots M_\ell\\ \epsilon_j\in\{-1,+1\}}}\int_{0<v_1,\dots,v_\ell<\infty}\hat{\psi}(\epsilon_1 v_1+\cdots+\epsilon_\ell v_\ell)$$

$$\prod_{j=1}^{\ell}\Big[\frac{1}{(2\pi i)^2}\iint_{\substack{|w_j-1|=\epsilon\\ |z_j-1|=\epsilon}} M_j^{-z_j}N_j^{s+1-w_j}\sum_{h_j,q_j}\frac{r_{q_j}(h_j)}{h_j^{s+2-w_j-z_j}}v_j^{s+1-w_j-z_j}$$

$$D_{A_j}(w_j,e(\frac{N_j}{q_j}))D_{B_j}(z_j,e(\frac{M_j}{q_j}))(\frac{T}{2\pi})^{w_j+z_j-2}dw_jdz_j\ dv_j\Big]\ \frac{ds}{s}$$

$$= \quad \frac{1}{2\pi i}\int_{(2)} X^s\int_0^\infty \psi(t)\sum_{\substack{U(\ell)\subset A\\ V(\ell)\subset B}}\left(\frac{Tt}{2\pi}\right)^{-\sum_{\substack{\hat\alpha\in U(\ell)\\ \hat\beta\in V(\ell)}}(\hat\alpha+\hat\beta+s)}$$

$$\times\mathcal{B}(A_s-U(\ell)_s+V(\ell)^-,B-V(\ell)+U(\ell)_s{}^-,1)\ dt\ \frac{ds}{s}$$

where $U(\ell)$ denotes a set of cardinality ℓ with precisely one element from each of $A_1,\dots,A_\ell$ and similarly $V(\ell)$ denotes a set of cardinality ℓ with precisely one element from each of $B_1,\dots,B_\ell$.

Automorphisms

If we sum this over all the ways to split up A and B we get what the recipe predicts times a factor

$$\ell!^2\,\ell^{2k-2\ell}$$

But this is the number of automorphisms of the *-system.

We can use this approach to discover a formula for $\gamma_k(c),\ c > 1.$

It is a linear combination of the functions

$$\int_{z,w} (\eta + \sum (z_j + w_j))^{k^2} \frac{\prod_j \left[(1-z_j)^k (1-w_j)^k\right] \Delta(z)^2 \Delta(w)^2}{\prod_j \left[z_j^k w_j^k\right] \prod_{i,j} \left[(1-w_i-z_j)(1+w_i+z_j)\right]} \ dz\ dw$$

where η is the fractional part of c
and the products are for $1 \le i, j \le c$

Wooley has pointed out the connection with counting points on varieties and Manin's idea of counting points on certain varieties by counting points on a stratified set of subvarieties; this idea may be relevant here.

Trevor Wooley

Y. I.Manin

Manin stratification

A classical example is given by equal sums of 3 fourth powers:

$$x_1^4 + x_2^4 + x_3^4 = y_1^4 + y_2^4 + y_3^4.$$

When one analyzes the major arcs, the obvious diagonal solutions $x_1 = y_1, x_2 = y_2, x_3 = y_3$ (and permutations) are missed! Also the solutions that arise from the subvariety where $x_1 = x_2 + x_3$ and $y_1 = y_2 + y_3$ are also missed. If one is counting solutions in a box with all variables bounded by N, the major arcs deliver an asymptotic of order N^2. The diagonal solutions give $\approx N^3$ solutions. And the solutions on the special subvariety mentioned above give $\approx N^2 \log N$ solutions. We see this story as explaining why Type-II terms are the right ones to calculate and the moral of the story is that these extra solutions are hiding in the minor arcs and need to be separately identified and teased out.

We can see an example already from the case k=2. We expect that

$$\sum_{n \le X} \frac{d(n)}{n^s} - \mathrm{Res}_{w=1-s} \frac{\zeta(s+w)^2 X^w}{w}$$

is a good approximation to zeta(s)^2 when X >>T^2.

But the analysis of

$$\sum_{\substack{m \le X \\ |h| \ge 1}} d(m)d(m+h)\hat{\psi}(hT/m)$$

averaged over h fails to reveal a large main term of size X/T^2
as well as a secondary main term that reflects the change in behavior
as X passes T^2.

There is a closer analogy between moments of zeta and averages of characteristic polynomials than just that the main terms agree once we insert the arithmetic factor.

In a shifted moment of zeta we let the arithmetic factors go to 1 and we replace zeta(1+x) by z(x). The shift alpha for zeta becomes exp(alpha) in RMT. Finally N becomes t/(2 pi). At that point ALL of the main terms agree.

It stands to reason that we can learn something by carefully analyzing all of the pieces from both points of view.

And, if we regard the matrix size N in RMT as the infinite prime it stands to reason that we should investigate carefully what happens with the finite primes as well.

GUE

The same circle of ideas works for averages of ratios of zeta-functions and characteristic polynomials.
In particular, using ratios, we can revisit the Bogomolny-Keating papers on

`Hardy-Littlewood implies GUE’

on a similar footing as this work.

RMT problems

1. Find (simple) exact formulas for $I_k(m;N)$ for m>N?

2. Do the $I_k(m;N)$ satisfy a Painleve?

3. What is the RMT version of CK V?

4. Does the answer to 3 lead to a recursion formula for $I_k(m;N)$?

5. Can we find the analogue of $I_k(m;N)$ for O(N) and USp(2N)?

6. Does 5 lead to a formulation of CK V for moments in other families?

Function field problems

1. Can we do pair correlation over function fields?

2. Can we reproduce any of KRRR for fixed q?

3. Analogue of CK I-V for function fields?

4. Analogue of divisors in short intervals and in arithmetic progressions
   for other families?

Number field problems

1. Do Rodgers-Soundararajan with shifts and with power savings.

2. What is the precise connection between the answer to 1 and the moment polynomials?

3. Let

$$\mathcal{M}_\lambda(X,T;W) := \sum_{\substack{m\le X\\ h\ge 1}} \frac{\lambda(m)\lambda(m+h)}{m} W\left(\frac{hT}{m}\right)$$

and

$$\mathcal{S}_\lambda(X;T) := \sum_{m\le X} \left|\sum_{h=1}^{H} \lambda(m+h)\right|^2$$

Work out the sizes (asymptotics?) and
the relationship between these averages for various lambda and
ranges of T and X. When X > T^2 is it best to average
over h first, possibly with Voronoi (see Jutila, Ivic)?
Connections with Manin stratification.)

4. Extend Nathan Ng's work to rigorously obtain averages of long Dirichlet polynomials with general divisor coefficients.

5. Asymptotics of 10th moment of L-functions in cusp form families

6. Rigorously derive n-correlation conjecture in a range [0,2] from H-L conjectures

FRG Project

A newly funded NSF project with PIs

Conrey, Iwaniec, Keating, Soundararajan, Wooley

and senior scientists

Brad Rodgers and Caroline Turnage-Butterbaugh

will be devoted to understanding these questions. If you are interested in this project send me email at

conrey@aimath.org

with subject line

FRG

The End